

# HOMOLOGICAL FLAT DIMENSIONS

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**ABSTRACT.** For finitely generated module  $M$  over a local ring  $R$ , the conventional notions of complete intersection dimension  $\text{CI-dim}_R M$  and Cohen-Macaulay dimension  $\text{CM-dim}_R M$  do not extend to cover the case of infinitely generated modules. In this paper we introduce similar invariants for not necessarily finitely generated modules, (namely, complete intersection flat and Cohen-Macaulay flat dimensions) which for finitely generated modules, coincide with the corresponding classical ones.

## 1. INTRODUCTION

An important motivation for studying homological dimensions goes back to 1956 when Auslander, Buchsbaum and Serre proved the following theorem: A commutative noetherian local ring  $R$  is regular if the residue field  $k$  has finite projective dimension and only if all  $R$ -modules have finite projective dimension. This introduced the theme that finiteness of a homological dimension for all modules singles out rings with special properties.

Auslander and Bridger [3], introduced a homological dimension designed to single out modules with properties similar to those of modules over Gorenstein rings. They called it G-dimension and it is a refinement of the projective dimension and showed that a local noetherian ring  $(R, \mathfrak{m}, k)$  is Gorenstein if the residue field  $k$  has finite G-dimension and only if all finitely generated  $R$ -modules have finite G-dimension. More recently, other homological dimensions have been introduced to characterize complete intersection and Cohen-Macaulay rings (see [11], [25] and [5] for an overview.)

This paper is concerned with homological dimensions for not necessarily *finitely generated* modules over commutative noetherian local rings  $(R, \mathfrak{m}, k)$  with identity. For any  $R$ -module  $M$ , the flat dimension of  $M$  over  $R$  is denoted by  $\text{fd}_R M$ . There is always an inequality  $\text{fd}_R M \leq \text{pd}_R M$ , and equality holds if  $M$  is *finite*, that is finitely generated, where  $\text{pd}_R M$  denotes for projective dimension of  $M$ . A deep result, due to Gruson-Raynaud [36] and Jensen [31], says that flat  $R$ -modules have finite projective dimension. Hence the flat dimension and the projective dimension of a module are finite simultaneously. Therefore it seems that, the flat dimension is a good and suitable extension of the projective dimension for non-finite modules.

In [15] Christensen, Foxby, and Frankild introduced the *large restricted flat dimension* which is denoted by  $\text{Rfd}$  and it is defined by the formula

$$\text{Rfd}_R M = \sup\{i | \text{Tor}_i^R(L, M) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{fd}_R L < \infty\}.$$

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2000 *Mathematics Subject Classification.* 13H10, 13C15, 13D05.

*Key words and phrases.* flat dimension, homological dimension, Auslander-Buchsbaum formula, intersection theorem.

T. Sharif was supported in part by a grant from IPM (No. 83130311).

S. Yassemi was supported in part by a grant from IPM (No. 861300000).

They showed that for all  $R$ -module  $M$ , there is an inequality

$$\text{Rfd}_R M \leq \text{fd}_R M$$

with equality if  $\text{fd}_R M < \infty$ .

In [19] and [20] Enochs and Jenda have introduced the Gorenstein flat dimension  $\text{Gfd}_R M$  of any  $R$ -module  $M$ . An  $R$ -module  $M$  is said to be Gorenstein flat if and only if there is an exact sequence

$$\cdots \rightarrow F^{-1} \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$$

of flat  $R$ -modules such that  $M = \ker(F^0 \rightarrow F^1)$  and such that for any injective  $R$ -module  $I$ ,  $I \otimes_R -$  preserves the exactness of the above complex. The Gorenstein flat dimension is defined by using Gorenstein flat modules in a fashion similar to that of flat dimension. Recall that for a finite  $R$ -module  $M$  we have  $\text{Gfd}_R M = \text{G-dim}_R M$  by [20]. Holm has studied this concept further in [27] and proved that  $\text{Gfd}_R M$  is a refinement of  $\text{fd}_R M$  and that  $\text{Rfd}_R M$  is a refinement of  $\text{Gfd}_R M$ . In other words, for any  $R$ -module  $M$  there is a chain of inequalities

$$\text{Rfd}_R M \leq \text{Gfd}_R M \leq \text{fd}_R M,$$

and if one of these quantities is finite then there is equality everywhere to its left.

The main goal of this paper is to introduce and study notions of complete intersection flat dimension ( $\text{CIfd}$ ) and Cohen-Macaulay flat dimension ( $\text{CMfd}$ ) as refinements of flat dimensions for every module  $M$  over a noetherian ring  $R$  (see Sections 3 and 4 for definitions).

A main result of this paper is the comparison of the Gorenstein flat and the complete intersection flat dimensions as given by the following theorem (see Theorem 4.5):

**Theorem A.** Let  $M$  be an  $R$ -module. Then there is an inequality

$$\text{Gfd}_R M \leq \text{CIfd}_R M$$

with equality if  $\text{CIfd}_R M$  is finite.

Viewing the above theorem, there is the following sequence of inequalities:

$$\text{Rfd}_R M \leq \text{CMfd}_R M \leq \text{Gfd}_R M \leq \text{CIfd}_R M \leq \text{fd}_R M.$$

If one of these dimensions is finite, then it is equal to those of its left.

We also introduce and study a variety of refinements of flat dimension, namely upper Cohen-Macaulay flat dimension ( $\text{CM}^*\text{fd}$ ) and upper Gorenstein flat dimension ( $\text{G}^*\text{fd}$ ) for every module  $M$  over a noetherian ring  $R$  (see Sections 3 for definitions). These dimensions fit into the following scheme of inequalities:

$$\text{Rfd}_R M \leq \text{CM}^*\text{fd}_R M \leq \text{G}^*\text{fd}_R M \leq \text{CIfd}_R M \leq \text{fd}_R M,$$

with equality to the left of any finite number.

The new homological flat dimensions are in many respects, similar to the classical ones. As a second example of what can be gained from our homological flat dimensions, we have the following result which is called Intersection Theorem for homological flat dimensions; (see Theorem 3.5):

**Theorem B.** Let  $M$  be an  $R$ -module, with  $\text{Hfd}_R M < \infty$  and of finite depth. Suppose that  $R$  is an equicharacteristic zero ring, then:

$$\dim R \leq \dim_R M + \text{Hfd}_R M,$$

for  $H = \text{CI}, G^*,$  and  $\text{CM}^*$ .

In Section 4, a number of base change results for homological flat dimensions are obtained (see Propositions 4.3, 4.7). Special attention is given to finite homomorphisms  $R \rightarrow R/(x_1, \dots, x_n)$  where  $x_1, \dots, x_n$  is an  $R$ -regular elements (see Propositions 4.9 and 4.10).

The Auslander-Buchsbaum formula asserts that if a finitely generated  $R$ -module  $M$  has finite projective dimension, then  $\text{depth}_R M + \text{pd}_R M = \text{depth } R$ . In [2] Auslander further generalized this formula for  $M$  as before and  $N$  a finitely generated  $R$ -module. In fact he showed that for  $s = \sup\{n | \text{Tor}_n^R(M, N) \neq 0\}$  if either  $s = 0$  or  $\text{depth}_R \text{Tor}_s^R(M, N) \leq 1$ , then

$$(*) \quad s = \text{depth } R - \text{depth}_R M - \text{depth}_R N + \text{depth}_R \text{Tor}_n^R(M, N).$$

More generally we say that the *depth formula* holds for  $M$  and  $N$  if  $s$  is finite and  $(*)$  holds. In Section 5 the following result which may be regarded as an analogue of Auslander's theorem for  $\text{Clfd}$  is proven; (see Theorem 5.2):

**Theorem C.** Let  $M$  and  $N$  be  $R$ -modules such that  $\text{Clfd}_R M < \infty$ . If  $s$  is finite, then

$$s \geq \text{depth } R - \text{depth}_R M - \text{depth}_R N$$

with equality if and only if  $\text{depth}_R \text{Tor}_s^R(M, N) = 0$ .

This is an extension (to non-finite case) of [32, Theorem (2.2)].

In Section 6 basic properties of homological flat dimensions for finitely generated modules are established and in Section 7 we discuss various homological injective dimensions for modules over noetherian rings namely, the Cohen-Macaulay injective dimension ( $\text{CMid}$ ), upper Cohen-Macaulay injective dimension ( $\text{CM}^*\text{id}$ ), upper Gorenstein injective dimension ( $\text{G}^*\text{id}$ ) and complete intersection injective dimension ( $\text{Clid}$ ). These dimensions satisfy the following inequalities

$$\text{Chid}_R M \leq \text{CMid}_R M \leq \text{CM}^*\text{id}_R M \leq \text{G}^*\text{id}_R M \leq \text{Clid}_R M \leq \text{id}_R M,$$

where

$$\text{Chid}_R M = \sup\{\text{depth } R_{\mathfrak{p}} - \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} | \mathfrak{p} \in \text{Spec}(R)\}.$$

Recall that  $\text{width}_R M = \inf\{i | \text{Tor}_i^R(M, k) \neq 0\}$ .

It is natural to ask when homological flat dimensions satisfy a formula of Auslander-Buchsbaum type. The answer is given in the following theorem; (see Theorem 8.4):

**Theorem D.** Let  $R$  be a Cohen-Macaulay local ring and let  $M$  be an  $R$ -module of finite  $\text{Hfd}_R M$  for  $H = \text{CI}, \text{G}^*, \text{CM}^*$ , and  $\text{CM}$ . Then  $\text{Hfd}_R M + \text{depth}_R M = \text{depth } R$  if and only if  $\text{depth}_R M \leq \text{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Supp}(M)$ .

## 2. DEFINITIONS AND NOTATIONS

In this section we recall various definitions of homological dimensions for finite modules.

**Definition 2.1.** A finite  $R$ -module  $M$  has  $G$ -dimension 0 if the following conditions are satisfied:

- (i)  $M \cong \text{Hom}_R(\text{Hom}_R(M, R), R)$ ,
- (ii)  $\text{Ext}_R^i(M, R) = 0$  for all  $i > 0$ , and
- (iii)  $\text{Ext}_R^i(\text{Hom}_R(M, R), R) = 0$  for all  $i > 0$ .

The Gorenstein dimension of  $M$  which is defined by Auslander and Bridger [3] and denoted by  $\text{G-dim}_R M$ , as the least number  $n$  for which there exists an exact sequence

$$0 \rightarrow G_n \rightarrow G_{n-1} \rightarrow \cdots \rightarrow G_0 \rightarrow M \rightarrow 0,$$

where  $G_i$  has  $G$ -dimension 0 for  $i = 0, \dots, n$ .

A finite  $R$ -module  $M$  is called *perfect* (resp. *G-perfect*) if  $\text{pd}_R M = \text{grade}_R M$  (resp.  $\text{G-dim}_R M = \text{grade}_R M$ ). Let  $Q$  be a local ring and  $J$  an ideal of  $Q$ . By abuse of language we say that  $J$  is *perfect* (resp. *G-perfect*) if the  $Q$ -module  $Q/J$  has the corresponding property.

The ideal  $J$  is called Gorenstein if it is perfect and  $\beta_g^Q(Q/J) = 1$  for  $g = \text{grade}_Q J$ , where  $\beta_g^Q(Q/J)$  stands for  $g$ -th betti number of  $Q/J$ . It is called complete intersection ideal, if  $J$  is generated by an  $R$ -regular elements.

We say that  $R$  has a CI-deformation (resp.  $G^*$ -deformation, CM-deformation) if there exists a local ring  $Q$  and a complete intersection (resp. Gorenstein, G-perfect) ideal  $J$  in  $Q$  such that  $R = Q/J$ . A CI-quasi-deformation (resp.  $G^*$ -quasi-deformation, CM-quasi-deformation) of  $R$  is a diagram of local homomorphisms  $R \rightarrow R' \leftarrow Q$ , with  $R \rightarrow R'$  a flat extension and  $R' \leftarrow Q$  a CI-deformation (resp.  $G^*$ -deformation, CM-deformation). We set  $M' = M \otimes_R R'$ .

The *complete intersection dimension* of  $M$  as defined by Avramov, Gasharov, and Peeva [11] and denoted by  $\text{CI-dim}_R M$  is

$$\text{CI-dim}_R M := \inf\{\text{pd}_Q M' - \text{pd}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a CI-quasi-deformation}\}.$$

The *upper Gorenstein dimension* of  $M$  as defined by Veliche [43] and denoted by  $\text{G}^*\text{-dim}_R M$  is

$$\text{G}^*\text{-dim}_R M := \inf\{\text{pd}_Q M' - \text{pd}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a } G^*\text{-quasi-deformation}\}.$$

The *Cohen-Macaulay dimension* of  $M$ , as defined by Gerko [25] and denoted by  $\text{CM-dim}_R M$  is

$$\text{CM-dim}_R M := \inf\{\text{G-dim}_Q M' - \text{G-dim}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a CM-quasi-deformation}\}.$$

There are the following sequence of inequalities:

$$\text{CM-dim}_R M \leq \text{G-dim}_R M \leq \text{G}^*\text{-dim}_R M \leq \text{CI-dim}_R M \leq \text{pd}_R M,$$

with equality to the left of any finite number.

In [18] Enochs and Jenda introduced the Gorenstein injective dimension  $\text{Gid}_R M$  of any  $R$ -module  $M$  as follows:

**Definition 2.2.** *An  $R$ -module  $M$  is said to be Gorenstein injective if and only if there is an exact sequence*

$$\cdots \rightarrow E^{-1} \rightarrow E^0 \rightarrow E^1 \rightarrow \cdots$$

*of injective  $R$ -modules such that  $M = \ker(E^0 \rightarrow E^1)$  and for any injective  $R$ -module  $E$ ,  $\text{Hom}_R(E, -)$  preserves exactness of the above complex. The Gorenstein injective dimension is defined by using Gorenstein injective modules in a fashion similar to that of injective dimension.*

It is known that  $\text{Gid}_R M \leq \text{id}_R M$  with equality if  $\text{id}_R M$  is finite.

For a noetherian ring the following categories were introduced by Avramov and Foxby [7]:

**Definition 2.3.** Let  $R$  be a ring with a dualizing complex  $D$ . Let  $\mathcal{D}_b(R)$  denote the full subcategory of  $\mathcal{D}(R)$  (the derived category of  $R$ -complexes) consisting of complexes  $X$  with  $H_n(X) = 0$  for  $n \gg 0$ . The Auslander class  $\mathbf{A}(R)$  is defined as the full subcategory of  $\mathcal{D}_b(R)$ , consisting of those complexes  $X$  for which  $D \otimes_R^{\mathbf{L}} X \in \mathcal{D}_b(R)$  and the canonical morphism

$$\gamma_X : X \rightarrow \mathbf{R}Hom_R(D, D \otimes_R^{\mathbf{L}} X),$$

is an isomorphism. The Bass class  $\mathbf{B}(R)$  is defined as the full subcategory of  $\mathcal{D}_b(R)$ , consisting of those complexes  $X$  for which  $\mathbf{R}Hom_R(D, X) \in \mathcal{D}_b(R)$  and the canonical morphism

$$\iota_X : D \otimes_R^{\mathbf{L}} \mathbf{R}Hom_R(D, X) \rightarrow X,$$

is an isomorphism.

**Remark 2.4.** It is proved in [16, (4.1) and (4.4)] that if  $R$  admits of a dualizing complex then for an  $R$ -module  $M$  we have:

- (a)  $M \in \mathbf{A}(R)$  if and only if  $\text{Gfd}_R M < \infty$ , and
- (b)  $M \in \mathbf{B}(R)$  if and only if  $\text{Gid}_R M < \infty$ .

See also [21] and [22] for an interesting extension of this result.

### 3. COMPLETE INTERSECTION FLAT DIMENSION

In this section we introduce *complete intersection flat dimension*, *upper Gorenstein flat dimension*, and *upper Cohen-Macaulay flat dimension* for not necessarily finite  $R$ -modules, and verify a number of their properties which are similar to those for the flat dimension.

We say that  $R$  has a  $\text{CM}^*$ -deformation if there exist a local ring  $Q$  and a perfect ideal  $J$  in  $Q$  such that  $R = Q/J$ . A  $\text{CM}^*$ -quasi-deformation of  $R$  is a diagram of local homomorphisms  $R \rightarrow R' \leftarrow Q$  with  $R \rightarrow R'$  a flat extension and  $R' \leftarrow Q$  a  $\text{CM}^*$ -deformation. We set  $M' = M \otimes_R R'$ .

**Definition 3.1.** Let  $M \neq 0$  be an  $R$ -module. The *complete intersection flat dimension*, *upper Gorenstein flat dimension*, and *upper Cohen-Macaulay flat dimension* of  $M$ , are defined as:

$$\text{CI}fd_R M := \inf\{fd_Q M' - fd_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a CI-quasi-deformation}\}$$

$$G^*fd_R M := \inf\{fd_Q M' - fd_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a } G^*\text{-quasi-deformation}\}$$

$\text{CM}^*fd_R M := \inf\{fd_Q M' - fd_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a } \text{CM}^*\text{-quasi-deformation}\}$ , respectively. We complement this by  $\text{Hfd}_R 0 = -\infty$  for  $H = \text{CI}$ ,  $G^*$ , and  $\text{CM}^*$ .

Our first result says that the large restricted flat dimension is a refinement of the above H-flat dimensions.

**Proposition 3.2.** Let  $R \rightarrow S \leftarrow Q$  be a  $\text{CM}$ -quasi-deformation, and let  $M$  be an  $R$ -module. Then

$$\text{Rfd}_Q(M \otimes_R S) - \text{Rfd}_Q S = \text{Rfd}_R M.$$

*Proof.* First we prove the equality

$$\text{Rfd}_S N + \text{G-dim}_Q S = \text{Rfd}_Q N,$$

for an  $S$ -module  $N$ . To this end, choose by [15, (2.4)(b)] a prime ideal  $\mathfrak{p}$  of  $S$  such that the first equality below holds. Let  $\mathfrak{q}$  be the inverse image of  $\mathfrak{p}$  in  $Q$ . Therefore

there is an isomorphism  $N_{\mathfrak{p}} \cong N_{\mathfrak{q}}$  of  $Q_{\mathfrak{q}}$ -modules and a CM-deformation  $Q_{\mathfrak{q}} \rightarrow S_{\mathfrak{p}}$ . Hence

$$\begin{aligned} \text{Rfd}_S N &= \text{depth } S_{\mathfrak{p}} - \text{depth}_{S_{\mathfrak{p}}} N_{\mathfrak{p}} \\ &= \text{depth}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \text{depth}_{Q_{\mathfrak{q}}} N_{\mathfrak{p}} \\ &= \text{depth } Q_{\mathfrak{q}} - \text{G-dim}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} - \text{depth}_{Q_{\mathfrak{q}}} N_{\mathfrak{p}} \\ &\leq \text{Rfd}_Q N - \text{G-dim}_{Q_{\mathfrak{q}}} S_{\mathfrak{p}} \\ &= \text{Rfd}_Q N - \text{G-dim}_Q S. \end{aligned}$$

The second equality holds since  $Q_{\mathfrak{q}} \rightarrow S_{\mathfrak{p}}$  is surjective; the third equality holds by Auslander-Bridger formula [3]; the fourth equality is due to the G-perfectness assumption of  $S$  over  $Q$ ; while the inequality follows from [15, (2.4)(b)]. Now by [42, (3.5)] we have

$$\text{Rfd}_Q N \leq \text{Rfd}_S N + \text{Rfd}_Q S \leq \text{Rfd}_Q N - \text{G-dim}_Q S + \text{Rfd}_Q S = \text{Rfd}_Q N,$$

which is the desired equality.

Now we have

$$\begin{aligned} \text{Rfd}_Q(M \otimes_R S) &\leq \text{Rfd}_S(M \otimes_R S) + \text{Rfd}_Q S \\ &= \text{Rfd}_S(M \otimes_R S) + \text{G-dim}_Q S \\ &= \text{Rfd}_Q(M \otimes_R S), \end{aligned}$$

where the inequality was proven in [42, (3.5)], the first equality follows from the hypotheses, and the last follows from the above observation. Hence

$$\text{Rfd}_Q(M \otimes_R S) - \text{Rfd}_Q S = \text{Rfd}_S(M \otimes_R S) = \text{Rfd}_R M$$

where the second equality holds by [30, (8.5)].  $\square$

**Theorem 3.3.** *Let  $M$  be an  $R$ -module. Then we have  $\text{Rfd}_R M \leq \text{Hfd}_R M$ , for  $H = CI$ ,  $G^*$ , and  $CM^*$ , with equality if  $\text{Hfd}_R M$  is finite. In this case we have*

$$\text{Hfd}_R M = \sup\{\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

*Proof.* The inequality follows easily from the definitions of various H-flat dimensions introduced above, and Proposition 3.2. The last equality follows from [15, (2.4)(b)].  $\square$

The following corollary is an immediate consequence of Theorem 3.3:

**Corollary 3.4.** *There is the following chain of inequalities:*

$$\text{Rfd}_R M \leq CM^* \text{fd}_R M \leq G^* \text{fd}_R M \leq CI \text{fd}_R M \leq \text{fd}_R M,$$

*with equality to the left of any finite number.*

In [23, (19.7)] Foxby proved an Intersection Theorem for flat dimension. More precisely, he showed that for  $M$  an  $R$ -module of finite flat dimension and of finite depth, and  $R$  admitting of a Hochster module (as is the case where  $R$  is equicharacteristic), one has:

$$\dim R \leq \dim_R M + \text{fd}_R M.$$

Recall that the local ring  $(R, \mathfrak{m}, k)$  is equicharacteristic if  $\text{char} R = \text{char} k$ , where  $\text{char} R$  denotes the characteristic of the ring  $R$ . Now we extend Foxby's result to the homological flat dimensions in the following theorem:

**Theorem 3.5.** *Let  $M$  be an  $R$ -module of finite depth such that  $\text{Hfd}_R M < \infty$ . Suppose that  $R$  is an equicharacteristic zero ring, then:*

$$\dim R \leq \dim_R M + \text{Hfd}_R M,$$

for  $H = CI$ ,  $G^*$ , and  $CM^*$ .

The proof of this theorem makes use of Lemma 3.6 below and the notion of the Cohen-Macaulay defect ( $\text{cmd}$ ) of a ring  $R$  which is defined as:

$$\text{cmd } R := \dim R - \text{depth } R.$$

**Lemma 3.6.** *Let  $Q \rightarrow R'$  be any  $CM^*$ -deformation. Then  $\text{cmd } R' \leq \text{cmd } Q$ .*

*Proof.* Suppose that  $J = \ker(Q \rightarrow R')$ . Since  $J$  is a perfect ideal of  $Q$ , we have  $\text{pd}_Q R' = \text{grade}_Q J$ . The proof of the Lemma is easily completed by noting that  $\text{depth } Q - \text{depth}_Q R' = \text{pd}_Q R' = \text{grade}_Q J \leq \text{ht } J \leq \dim Q - \dim R'$ , in which the first equality follows the Auslander-Buchsbaum formula.  $\square$

*Proof of Theorem 3.5.* It is sufficient to prove the Theorem for  $H = CM^*$ . Choose a  $CM^*$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$  such that  $\text{fd}_Q(M \otimes_R R') < \infty$  and  $CM^* \text{fd}_R M = \text{fd}_Q(M \otimes_R R') - \text{fd}_Q R'$ . It can be seen that  $Q$  is an equicharacteristic zero ring. Since  $R \rightarrow R'$  is a flat extension and  $\text{depth}_R M < \infty$ , it follows from [29, (2.6)] that  $\text{depth}_{R'}(M \otimes_R R') < \infty$ . Therefore we obtain  $\text{depth}_Q(M \otimes_R R') < \infty$  since  $Q \rightarrow R'$  is surjective. By Lemma 3.6 there is an inequality  $\text{cmd } R' \leq \text{cmd } Q$ . Then  $\text{cmd } R + \text{cmd } R'/\mathfrak{m}R' \leq \text{cmd } Q$ . So we have:

$$\begin{aligned} \dim R &\leq \text{cmd } Q - \text{cmd } R'/\mathfrak{m}R' + \text{depth } R \\ &= \dim Q - \dim R'/\mathfrak{m}R' - \text{depth } Q + \text{depth } R + \text{depth } R'/\mathfrak{m}R' \\ &= \dim Q - \dim R'/\mathfrak{m}R' - \text{depth } Q + \text{depth } R' \\ &= \dim Q - \dim R'/\mathfrak{m}R' - \text{pd}_Q R' \\ &\leq \dim_Q(M \otimes_R R') + \text{fd}_Q(M \otimes_R R') - \text{fd}_Q R' - \dim R'/\mathfrak{m}R' \\ &= \dim_{R'}(M \otimes_R R') + CM^* \text{fd}_R M - \dim R'/\mathfrak{m}R' \\ &= \dim_R M + CM^* \text{fd}_R M, \end{aligned}$$

where the third equality holds by the Auslander-Buchsbaum formula; and the second inequality holds from Foxby's Theorem [23, (19.7)]. To prove the fifth equality assume that  $M$  is the direct union of finite submodules  $M_i$  of  $M$  (for  $i$  in a directed set  $I$ ). Then

$$\dim_R M = \sup\{\dim_R M_i | i \in I\}.$$

So we get that  $M \otimes_R R'$  is the direct union of  $M_i \otimes_R R'$ . Consequently by the above observation we have:

$$\begin{aligned} \dim_{R'}(M \otimes_R R') &= \sup\{\dim_{R'}(M_i \otimes_R R') | i \in I\} \\ &= \sup\{\dim_R M_i + \dim R'/\mathfrak{m}R' | i \in I\} \\ &= \sup\{\dim_R M_i | i \in I\} + \dim R'/\mathfrak{m}R' \\ &= \dim_R M + \dim R'/\mathfrak{m}R', \end{aligned}$$

where the second equality follows from [12, (A.11)].  $\square$

#### 4. COHEN-MACAULAY FLAT DIMENSION

In this section we introduce the notion of *Cohen-Macaulay flat dimension* denoted by  $\text{CMfd}$ . For a finite  $R$ -module  $M$  it coincides with the Cohen-Macaulay dimension  $\text{CM-dim}_R M$  of Gerko. And we show that, for an  $R$ -module  $M$  we have the following sequence of inequalities

$$\text{Rfd}_R M \leq \text{CMfd}_R M \leq \text{Gfd}_R M \leq \text{Clfd}_R M \leq \text{fd}_R M,$$

with equality to the left of any finite number.

**Definition 4.1.** Let  $M \neq 0$  be an  $R$ -module. The *Cohen-Macaulay flat dimension* of  $M$ , is defined as:

$$\text{CMfd}_R M := \inf\{\text{Gfd}_Q M' - \text{Gfd}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a CM-quasi-deformation}\}.$$

We complement this by  $\text{CMfd}_R 0 = -\infty$ .

**Remark 4.2.** By taking the trivial CM-quasi-deformation  $R \rightarrow R \leftarrow R$ , one has  $\text{CMfd}_R M \leq \text{Gfd}_R M$ , and using Proposition 3.2 we have, when  $\text{CMfd}_R M < \infty$ , then  $\text{CMfd}_R M = \text{Rfd}_R M$ .

Notice that there is a notion of Cohen-Macaulay flat dimension in [28] which is different with ours. Before proceeding any further it is necessary to investigate the effect of change of ring on various notions of homological flat dimensions.

**Proposition 4.3.** Let  $M$  be an  $R$ -module. Let  $R \rightarrow R'$  be a local flat extension, and  $M' = M \otimes_R R'$ . Then

$$\text{Hfd}_R M \leq \text{Hfd}_{R'} M'$$

with equality when  $\text{Hfd}_{R'} M'$  is finite, for  $H = \text{CI}$ ,  $G^*$ ,  $\text{CM}^*$ , and  $\text{CM}$ .

*Proof.* We prove the result for Cohen-Macaulay flat dimension and the proof of the other cases are similar to this one, so we omit them. Suppose that  $\text{CMfd}_{R'} M' < \infty$ , and let  $R' \rightarrow R'' \leftarrow Q$  be a CM-quasi-deformation with  $\text{Gfd}_Q M'' < \infty$ , where  $M'' = M' \otimes_{R'} R''$ . Since  $R \rightarrow R'$  and  $R' \rightarrow R''$  are flat extensions, the local homomorphism  $R \rightarrow R''$  is also flat. Hence  $R \rightarrow R'' \leftarrow Q$  is a CM-quasi-deformation with  $\text{Gfd}_Q (M \otimes_R R'') < \infty$ . It follows that  $\text{CMfd}_R M$  is finite. Now by Theorem 3.3 and [30, (8.5)], we have

$$\text{CMfd}_R M = \text{Rfd}_R M = \text{Rfd}_{R'} M' = \text{CMfd}_{R'} M'.$$

$\square$



**Proposition 4.4.** *Let  $\widehat{R}$  be the completion of  $R$  relative to the  $\mathfrak{m}$ -adic topology. Then*

$$Hfd_R M = Hfd_{\widehat{R}}(M \otimes_R \widehat{R}),$$

for  $H = CI, G^*,$  and  $CM^*$ .

*Proof.* We prove the result for  $CM^*fd$  and the proof of the other cases are similar to this one. If  $CM^*fd_R M = \infty$ , then we obtain that  $CM^*fd_{\widehat{R}}(M \otimes_R \widehat{R}) = \infty$  by 4.3. Now assume that  $CM^*fd_R M < \infty$ . It is sufficient to prove that  $CM^*fd_{\widehat{R}}(M \otimes_R \widehat{R})$  is finite. Because in this case we have

$$CM^*fd_R M = Rfd_R M = Rfd_{\widehat{R}}(M \otimes_R \widehat{R}) = CM^*fd_{\widehat{R}}(M \otimes_R \widehat{R}),$$

in which the first and the last equalities follow from Theorem 3.3, and the middle one follows from [30, (8.5)].

For a  $CM^*$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$  of  $R$ , we have  $\widehat{R} \rightarrow \widehat{R'} \leftarrow \widehat{Q}$  is a  $CM^*$ -quasi-deformation of  $\widehat{R}$  with respect to their maximal ideal-adic completions. Now the equalities

$$\begin{aligned} fd_Q(M \otimes_R R') &= fd_{\widehat{Q}}(M \otimes_R R' \otimes_Q \widehat{Q}) = fd_{\widehat{Q}}(M \otimes_R \widehat{R'}) \\ &= fd_{\widehat{Q}}(M \otimes_R (\widehat{R} \otimes_{\widehat{R}} \widehat{R'})) = fd_{\widehat{Q}}((M \otimes_R \widehat{R}) \otimes_{\widehat{R}} \widehat{R'}), \end{aligned}$$

show that  $fd_{\widehat{Q}}((M \otimes_R \widehat{R}) \otimes_{\widehat{R}} \widehat{R'})$  is finite which imply that  $CM^*fd_{\widehat{R}}(M \otimes_R \widehat{R})$  is finite.  $\square$

One of the main result of this paper is Theorem 4.5 below the proof of which strongly makes use results of Sather-Wagstaff [40, Theorem F] and Esmkhani and Tousi [21, Corollary 2.6].

**Theorem 4.5.** *Let  $M$  be an  $R$ -module. Then there is the inequality*

$$Gfd_R M \leq CI fd_R M$$

with equality if  $CI fd_R M$  is a finite number.

*Proof. Step 1.* Assume that  $R$  admits of a dualizing complex  $D$ . We can actually assume that  $CI fd_R M$  is finite. So that by [40, Theorem F], there exists a CI-quasi-deformation  $R \rightarrow R' \leftarrow Q$  such that  $Q$  is complete, the closed fibre  $R'/\mathfrak{m}R'$  is artinian and Gorenstein, and  $fd_Q(M \otimes_R R')$  is finite. Therefore by Remark 2.4(a),  $M \otimes_R R'$  belongs to the the Auslander class  $\mathbf{A}(Q)$ . On the other hand since the kernel of  $Q \rightarrow R'$  is generated by  $Q$ -regular elements, using [8, Proposition 4.3] we deduce that it is a Gorenstein local homomorphism. Thus thanks to [7, Corollary (7.9)] we see that  $M \otimes_R R'$  belongs to the the Auslander class  $\mathbf{A}(R')$ . Note that  $R'$  is a complete local ring, so it admits of a dualizing complex. Hence using Remark 2.4(a), we obtain that  $Gfd_{R'}(M \otimes_R R')$  is finite. Since  $R'/\mathfrak{m}R'$  is a Gorenstein local ring, by [8, Proposition 4.2], we have  $R \rightarrow R'$  is a Gorenstein local homomorphism. Therefore by [8, Theorem 5.1], the complex  $D \otimes_R^{\mathbf{L}} R'$  is a dualizing complex of  $R'$ . Consequently by [16, Theorem 5.3],  $Gfd_R M$  is finite. Hence the equalities  $Gfd_R M = Rfd_R M = CI fd_R M$  hold.

**Step 2.** Now let  $R$  be any ring. Note that by Proposition 4.4 we have

$$CI fd_R M = CI fd_{\widehat{R}}(M \otimes_R \widehat{R}).$$

Since  $\widehat{R}$  admits of a dualizing complex by Step 1 we have

$$\text{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R}) \leq \text{Clfd}_{\widehat{R}}(M \otimes_R \widehat{R})$$

with equality if  $\text{Clfd}_{\widehat{R}}(M \otimes_R \widehat{R})$  is finite. Now assume that  $\text{Clfd}_R M$  is finite. Therefore  $\text{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R})$  is finite. Consequently by [21, Corollary 2.6],  $\text{Gfd}_R M$  is finite and  $\text{Gfd}_R M = \text{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R})$ . Hence

$$\text{Gfd}_R M = \text{Gfd}_{\widehat{R}}(M \otimes_R \widehat{R}) = \text{Clfd}_{\widehat{R}}(M \otimes_R \widehat{R}) = \text{Clfd}_R M.$$

This completes the proof.  $\square$

**Corollary 4.6.** *Let  $M$  be an  $R$ -module. Then there is the following sequence of inequalities*

$$\text{Rfd}_R M \leq \text{CMfd}_R M \leq \text{Gfd}_R M \leq \text{Clfd}_R M \leq \text{fd}_R M,$$

*with equality to the left of any finite number.*

**Proposition 4.7.** *Let  $M$  be an  $R$ -module. For each prime ideal  $\mathfrak{p} \in \text{Spec}(R)$  there is an inequality*

$$\text{Hfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \text{Hfd}_R M,$$

*for  $H = \text{CI}$ ,  $G^*$ ,  $\text{CM}^*$ , and  $\text{CM}$ .*

*Proof.* We prove the result for  $\text{CMfd}_R M$ , and the proof of the other cases are similar to this one. Choose a prime ideal  $\mathfrak{p} \in \text{Spec}(R)$ . Assume that  $\text{CMfd}_R M < \infty$  and fix a CM-quasi-deformation  $R \rightarrow R' \leftarrow Q$  such that  $\text{Gfd}_Q M' < \infty$ , where  $M' = M \otimes_R R'$ . Since  $R \rightarrow R'$  is faithfully flat extension of rings, there is a prime ideal  $\mathfrak{p}'$  in  $R'$  lying over  $\mathfrak{p}$ . Let  $\mathfrak{q}$  be the inverse image of  $\mathfrak{p}'$  in  $Q$ . The map  $R_{\mathfrak{p}} \rightarrow R'_{\mathfrak{p}'}$  is flat, and  $R'_{\mathfrak{p}'} \leftarrow Q_{\mathfrak{q}}$  is a CM-deformation. Therefore the diagram  $R_{\mathfrak{p}} \rightarrow R'_{\mathfrak{p}'} \leftarrow Q_{\mathfrak{q}}$  is a CM-quasi-deformation with  $\text{Gfd}_{Q_{\mathfrak{q}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} R'_{\mathfrak{p}'}) = \text{Gfd}_{Q_{\mathfrak{q}}} M'_{\mathfrak{q}'} \leq \text{Gfd}_Q M' < \infty$ . Hence  $\text{CMfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty$ . So we obtain

$$\begin{aligned} \text{CMfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} &= \text{Rfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \text{Rfd}_R M \\ &= \text{CMfd}_R M, \end{aligned}$$

in which the inequality holds by [15, (2.3)]. Thus the desired inequality follows.  $\square$

**Lemma 4.8.** *Let  $Q$  be a local ring, and let  $J \subseteq I$  be ideals of  $Q$ . Set  $R = Q/J$ . If  $J$  and  $I/J$  are perfect ideals of  $Q$  and  $R$  respectively, then  $I$  is a perfect ideal in  $Q$ .*

*Proof.* Since  $\text{pd}_Q R < \infty$  and  $\text{pd}_R Q/I < \infty$ , by [4, (3.8)] there is an equality  $\text{pd}_Q Q/I = \text{pd}_R Q/I + \text{pd}_Q R$ . By our assumption  $J$  is a perfect ideal of  $Q$  hence by [6, (2.7)] we have  $\text{grade}_Q Q/I = \text{grade}_R Q/I + \text{grade}_Q R$ . Using the perfectness of  $J$  in  $Q$  and  $I/J$  in  $R$ , we see that  $I$  is a perfect ideal in  $Q$ .  $\square$

**Proposition 4.9.** *Let  $\mathbf{x} = x_1, \dots, x_n$  be a sequence of elements of  $\mathfrak{m}$ , constituting  $R$ - and  $M$ -regular elements. Set  $\overline{R} = R/(\mathbf{x})$  and  $\overline{M} = M/(\mathbf{x})M$ . Then there are inequalities*

$$\begin{aligned} \text{Hfd}_{\overline{R}} \overline{M} &\leq \text{Hfd}_R M, \text{ and} \\ \text{Hfd}_R \overline{M} &\leq \text{Hfd}_R M + n, \end{aligned}$$

*with equality when,  $\text{Hfd}_R M$  is finite, for  $H = \text{CI}$ ,  $G^*$ , and  $\text{CM}^*$ .*

*Proof.* Since the proof for  $H = CI$  and  $G^*$  is analogous to  $CM^*$ , we only prove the proposition for  $H = CM^*$ . It is sufficient to prove the proposition for  $\mathbf{x} = x$  with  $x$  an  $R$ -regular and  $M$ -regular element. We may assume that  $CM^*fd_R M < \infty$  and choose a  $CM^*$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$  with  $fd_R M' < \infty$ , where  $M' = M \otimes_R R'$ . Thus  $R' = Q/J$ , where  $J$  is a perfect ideal of  $Q$ . We construct a  $CM^*$ -quasi-deformation of  $\overline{R}$ . Choose  $y \in Q$  mapping to  $x \in R'$ . Since  $x$  is  $R$ -regular, it is also  $R'$ -regular due to flatness of  $R'$  as an  $R$ -module. Set  $I = (y) + J$  and note that  $I/J = xR'$  is a perfect ideal of  $R'$ . Therefore by lemma 4.8,  $I$  is a perfect ideal in  $Q$  (for the case  $H = G^*$  use [43, (2.11)]). Set  $\overline{R'} = Q/I$ , and note that  $\overline{R} \rightarrow \overline{R'}$  is flat because  $R \rightarrow R'$  is flat. Thus  $\overline{R} \rightarrow \overline{R'} \leftarrow Q$  is a  $CM^*$ -quasi-deformation of  $\overline{R}$ .

Now we show that  $fd_Q(\overline{M} \otimes_{\overline{R}} \overline{R'})$  and  $fd_Q(\overline{M} \otimes_R R')$  are finite. We have the following isomorphisms

$$\overline{M} \otimes_{\overline{R}} \overline{R'} \cong \overline{M} \otimes_R \overline{R} \otimes_R R' \cong \overline{M} \otimes_R R'.$$

Since  $x$  is  $M$ -regular and  $R \rightarrow R'$  is flat, the exact sequence  $0 \rightarrow M \xrightarrow{x} M \rightarrow \overline{M} \rightarrow 0$  induces an exact sequence  $0 \rightarrow M' \xrightarrow{x} M' \rightarrow \overline{M} \otimes_R R' \rightarrow 0$ . So we obtain  $\overline{M} \otimes_R R' \cong M'/xM'$  and we have  $fd_Q(M'/xM') = fd_Q M' + 1$ . Hence we get  $CM^*fd_{\overline{R}} \overline{M}$  and  $CM^*fd_R \overline{M}$  are finite. Now the equalities

$$CM^*fd_{\overline{R}} \overline{M} = Rfd_{\overline{R}} \overline{M} = Rfd_R M = CM^*fd_R M,$$

where the second equality follows from [39, (3.11)], complete the proof of the first inequality in the assertion of the Theorem. The equalities

$$CM^*fd_R \overline{M} = Rfd_R \overline{M} = Rfd_{\overline{R}} \overline{M} + 1 = CM^*fd_R M + 1,$$

where the second equality follows from [42, (3.6)] and the third one holds by [39, (3.11)] complete the proof.  $\square$

**Proposition 4.10.** *Let  $\mathbf{x} = x_1, \dots, x_n$  be a  $R$ -regular elements. Set  $\overline{R} = R/(\mathbf{x})$ . For an  $\overline{R}$ -module  $M$ , then there is the inequality*

$$n + CI\text{fd}_{\overline{R}} M \leq CI\text{fd}_R M,$$

*with equality when  $CI\text{fd}_R M$  is finite.*

*Proof.* As usual we may assume that  $CI\text{fd}_R M < \infty$  and choose a  $CI$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$  with  $fd_Q M' < \infty$ , where  $M' = M \otimes_R R'$ . Consider  $\overline{R} \rightarrow \overline{R} \leftarrow R$  as a  $CI$ -quasi-deformation. One checks readily that  $\overline{R} \rightarrow R'' = \overline{R} \otimes_R R' \leftarrow Q$  is a  $CI$ -quasi-deformation of  $\overline{R}$ . From the equalities  $M \otimes_R R' = (M \otimes_{\overline{R}} \overline{R}) \otimes_R R' = M \otimes_{\overline{R}} (\overline{R} \otimes_R R') = M \otimes_{\overline{R}} R''$ , we obtain that  $fd_Q(M \otimes_R R'')$  and so  $CI\text{fd}_{\overline{R}} M$  are finite. Now the equalities

$$n + CI\text{fd}_{\overline{R}} M = n + Rfd_{\overline{R}} M = Rfd_R M = CI\text{fd}_R M,$$

where the second one holds by [42, (3.6)] complete the proof.  $\square$

Let  $\varphi : R \rightarrow S$  be a local homomorphism of complete local rings. Let  $N$  be a finite  $S$ -module, and let  $R \rightarrow R' \rightarrow S$  be a Cohen factorization of  $\varphi$  (cf. [10]). The following inequalities hold:

$$fd_R N \leq pd_{R'} N \leq fd_R N + \text{edim}(R'/\mathfrak{m}R')$$

$$Gfd_R N \leq G\text{-dim}_{R'} N \leq Gfd_R N + \text{edim}(R'/\mathfrak{m}R'),$$

where  $\text{edim}(R'/\mathfrak{m}R')$  is the minimal number of generators of the maximal ideal of  $R'/\mathfrak{m}R'$ . The first inequality is by [9] and the latter uses the recent characterization by Christensen, Frankild, and Holm of certain Auslander categories in terms of finiteness of G-dimensions (cf. [16], and also [30, Theorem 8.2]).

**Question 4.11.** *Let  $\varphi : R \rightarrow S$  be a local homomorphism of complete local rings. Let  $N$  be a finite  $S$ -module and let  $R \rightarrow R' \rightarrow S$  be a Cohen factorization of  $\varphi$ . The question is whether the following inequalities hold:*

$$\text{CI}fd_R N \leq \text{CI-dim}_{R'} N \leq \text{CI}fd_R N + \text{edim}(R'/\mathfrak{m}R').$$

## 5. THE DEPTH FORMULA

The point of this section is to prove the *depth formula* and note its immediate consequences.

**Notation 5.1.** *For  $R$ -modules  $M$  and  $N$  set*

$$fd_R(M, N) = \sup\{i \mid \text{Tor}_i^R(M, N) \neq 0\}.$$

*In particular, if  $\text{Tor}_n^R(M, N) = 0$  for all  $n$ , then  $fd_R(M, N) = -\infty$ , else  $0 \leq fd_R(M, N) \leq \infty$ . For a finite  $R$ -module  $M$ ,  $fd_R(M, k)$  is the usual flat dimension of  $M$  which is also equal to its projective dimension  $pd_R M$ . Moreover for such an  $M$ ,  $fd_R(M, N)$  is finite for every finitely generated  $N$ .*

**Theorem 5.2.** *Let  $M$  and  $N$  be  $R$ -modules such that  $\text{CI}fd_R M < \infty$ . If  $fd_R(M, N) < \infty$ , then*

$$fd_R(M, N) \geq \text{depth } R - \text{depth}_R M - \text{depth}_R N$$

*with equality if and only if  $\text{depth}_R \text{Tor}_s^R(M, N) = 0$ , for  $s = fd_R(M, N)$ .*

*Proof.* Since  $\text{CI}fd_R M < \infty$  there is, say a codimension  $c$  CI-quasi-deformation  $R \rightarrow R' \leftarrow Q$ , such that  $fd_Q M' < \infty$ , where  $M' = M \otimes_R R'$ . By codimension  $c$  we mean that the kernel of the homomorphism  $Q \rightarrow R'$  is generated by regular elements of length  $c$ . Choose  $\mathfrak{p} \in \text{Spec}(R')$  such that it is a minimal prime ideal containing  $\mathfrak{m}R'$ . Thus  $\mathfrak{m} = \mathfrak{p} \cap R$  and  $\mathfrak{p} = \mathfrak{q}/(\mathbf{x})$  for some  $\mathfrak{q} \in \text{Spec}(Q)$ , where  $(\mathbf{x}) = \ker(Q \rightarrow R')$ . Now the diagram  $R \rightarrow R'_\mathfrak{p} \leftarrow Q_\mathfrak{q}$  is a CI-quasi-deformation of the same codimension as  $R \rightarrow R' \leftarrow Q$ . It is clear that  $pd_Q R' = pd_{Q_\mathfrak{q}} R'_\mathfrak{p}$ . Also we have

$$fd_{Q_\mathfrak{q}}(M \otimes_R R'_\mathfrak{p}) = fd_{Q_\mathfrak{q}}(M \otimes_R (R' \otimes_Q Q_\mathfrak{q})) = fd_{Q_\mathfrak{q}}((M \otimes_R R') \otimes_Q Q_\mathfrak{q}) \leq fd_Q M' < \infty.$$

Hence  $\text{CI}fd_R M \leq fd_{Q_\mathfrak{q}}(M \otimes_R R'_\mathfrak{p}) - fd_{Q_\mathfrak{q}} R'_\mathfrak{p}$ . Therefore we showed that complete intersection flat dimension can be computed from CI-quasi-deformations  $R \rightarrow R' \leftarrow Q$  such that the closed fiber  $R'/\mathfrak{m}R'$  is artinian.

Due to faithful flatness of  $R'$  we have the following equalities in which  $N' = N \otimes_R R'$

$$s = fd_R(M, N) = fd_{R'}(M', N').$$

Assume that  $c = 1$ . Consider the change of rings spectral sequence

$$\text{Tor}_p^{R'}(M', \text{Tor}_q^Q(R', N')) \Rightarrow \text{Tor}_{p+q}^Q(M', N').$$

If  $q > 1$ , then  $\text{Tor}_q^Q(R', N') = 0$  and for  $q \leq 1$   $\text{Tor}_q^Q(R', N') = N'$ . Now the above spectral sequence generates the following long exact sequence

$$\cdots \rightarrow \text{Tor}_{i+1}^{R'}(M', N') \rightarrow \text{Tor}_i^{R'}(M', N') \rightarrow \text{Tor}_i^Q(M', N') \rightarrow \text{Tor}_i^{R'}(M', N') \rightarrow \cdots$$

Therefore  $\text{Tor}_{s+1}^Q(M', N') = \text{Tor}_s^{R'}(M', N')$ . Iterating in the same manner we have

$$\text{Tor}_s^{R'}(M', N') = \text{Tor}_{s+c}^Q(M', N').$$

So  $\sup\{i \mid \text{Tor}_i^Q(M', N') \neq 0\} = s + c$ . Since  $\text{depth}(R'/\mathfrak{m}R') = 0$  and  $Q \rightarrow R'$  is surjective, the following equalities hold:

$$\text{depth}_Q \text{Tor}_s^{R'}(M', N') = \text{depth}_{R'} \text{Tor}_s^{R'}(M', N') = \text{depth}_R \text{Tor}_s^R(M, N),$$

and they are equal to  $\text{depth}_Q \text{Tor}_{s+c}^Q(M', N')$ . Since  $\text{fd}_Q M' < \infty$  it follows from [42, (2.3)] that

$$\begin{aligned} s + c &\geq \text{depth } Q - \text{depth}_Q M' - \text{depth}_Q N' \\ &= \text{depth } R + c - \text{depth}_R M - \text{depth}_R N, \end{aligned}$$

and equality holds if and only if  $\text{depth}_Q \text{Tor}_{s+c}^Q(M', N') = 0$ . Thus

$$s = \text{fd}_R(M, N) \geq \text{depth } R - \text{depth}_R M - \text{depth}_R N,$$

with equality if and only if  $\text{depth}_R \text{Tor}_s^R(M, N) = 0$ .  $\square$

**Definition 5.3.** We say that  $M$  and  $N$  satisfy the dependency formula over  $R$ , if  $\text{fd}_R(M, N) = \sup\{\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Supp}(M) \cap \text{Supp}(N)\}$ .

**Corollary 5.4.** Let  $M$  and  $N$  be  $R$ -modules such that  $\text{CI} \text{fd}_R M < \infty$ . If  $\text{fd}_R(M, N) < \infty$ , then  $M$  and  $N$  satisfy the dependency formula.

*Proof.* It is easy to see that:

$$\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \leq \text{fd}_R(M, N).$$

Using Theorem 5.2 we have  $\text{fd}_R(M, N) = \text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$  if and only if

$$\text{depth}_{R_{\mathfrak{p}}} \text{Tor}_s^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0,$$

or equivalently if and only if  $\mathfrak{p} \in \text{Ass}(\text{Tor}_s^R(M, N))$  for  $s = \text{fd}_R(M, N) < \infty$ .  $\square$

In Theorem 4.5 we proved that  $\text{Gfd}_R M \leq \text{CI} \text{fd}_R M$  for any  $R$ -module  $M$ . So it is natural to look for a dependency formula for Gorenstein flat dimension. In the following proposition we prove a dependency formula for Gorenstein flat dimension:

**Proposition 5.5.** Let  $M$  and  $N$  be  $R$ -modules, such that  $\text{Gfd}_R M < \infty$  and  $\text{id}_R N < \infty$ . Then  $M$  and  $N$  satisfy the dependency formula.

*Proof.* It is clear that  $\text{Gfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$  and  $\text{id}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$  are finite numbers, and we have  $\text{fd}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \text{fd}_R(M, N)$  for  $\mathfrak{p} \in \text{Supp}(M) \cap \text{Supp}(N)$ . Now from [23, (12.26)] and [17, (4.4)(a)] we get:

$$\text{depth } R_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \text{depth}_{R_{\mathfrak{p}}} N_{\mathfrak{p}} \leq \text{fd}_R(M, N),$$

with equality when  $\mathfrak{p} \in \text{Ass}(\text{Tor}_s^R(M, N))$ , for  $s = \text{fd}_R(M, N)$ .  $\square$

The following corollary due to Iyengar and Sather-Wagstaff [30, Theorem (8.7)] is an immediate consequence of the above proposition.

**Corollary 5.6.** Let  $M$  be an  $R$ -module such that  $\text{Gfd}_R M < \infty$ . Then

$$\sup\{i \mid \text{Tor}_i^R(M, E_R(k)) \neq 0\} = \text{depth } R - \text{depth}_R M,$$

where  $E_R(k)$  denotes to the injective envelope of  $k$ .

**Corollary 5.7.** *Let  $(R, \mathfrak{m})$  be a complete local ring and  $N$  and  $M$  be  $R$ -modules with  $M$  finite.*

(a) *If  $Gfd_R N < \infty$ , and  $fd_R M < \infty$ , then*

$$\sup\{i | Ext_R^i(N, M) \neq 0\} = depth R - depth_R N.$$

(b) *If  $CI fd_R N < \infty$ , then*

$$\sup\{i | Ext_R^i(N, M) \neq 0\} = depth R - depth_R N,$$

*provided that the left hand side is a finite number.*

*Proof.* (a) Set  $E = E_R(k)$ , the injective envelope of  $k$ . Since  $R$  is complete we have  $R = \text{Hom}_R(E, E)$ . Therefore we have

$$\begin{aligned} Ext_R^i(N, M) &\cong Ext_R^i(N, M \otimes_R R) \cong Ext_R^i(N, M \otimes_R \text{Hom}_R(E, E)) \\ &\cong Ext_R^i(N, \text{Hom}_R(\text{Hom}_R(M, E), E)) \\ &\cong \text{Hom}_R(\text{Tor}_i^R(N, \text{Hom}_R(M, E)), E). \end{aligned}$$

Consequently we have

$$\sup\{i | Ext_R^i(N, M) \neq 0\} = fd_R(N, \text{Hom}_R(M, E)).$$

Since  $fd_R M < \infty$  we have  $id_R \text{Hom}_R(M, E) < \infty$ . Now Proposition 5.5, gives the result.

(b) Similarly to that of part (a) one has

$$\sup\{i | Ext_R^i(N, M) \neq 0\} = fd_R(N, \text{Hom}_R(M, E)),$$

which is equal to  $depth R - depth_R N$  by Corollary 5.4 and [45, Lemma 2.2].  $\square$

The following example shows that the completeness assumption of  $R$  is crucial in part (a) of the above corollary.

**Example 5.8.** *Let  $(R, \mathfrak{m})$  be a local domain which is not complete with respect to the  $\mathfrak{m}$ -adic topology. In [1, (3.3)] it is shown that  $\text{Hom}_R(\widehat{R}, R) = 0$ . Therefore, when  $N = \widehat{R}$  and  $M = R$ . It is clear that the right hand side of the first equality equal to zero which is not equal to the left hand side.*

**Corollary 5.9.** *Let  $M$  and  $N$  be  $R$ -modules;*

(a) *If  $CI fd_R M < \infty$  then the following are equivalent:*

- (i)  $\text{Tor}_n^R(N, M) = 0 \ n \gg 0$ .
- (ii)  $\text{Tor}_n^R(N, M) = 0 \ n > CI fd_R M$ .

(b) *If  $R$  is a complete local ring and  $CI fd_R N < \infty$ , and  $M$  a finite  $R$ -module, then the following are equivalent:*

- (i)  $Ext_R^n(N, M) = 0 \ n \gg 0$ .
- (ii)  $Ext_R^n(N, M) = 0 \ n > depth R - depth_R M$ .

*Proof.* (a) If for all integer  $n$ ,  $\text{Tor}_n^R(M, N) = 0$ , then the assertion holds. So assume for some integer  $\ell$ ,  $\text{Tor}_\ell^R(M, N) \neq 0$ . Therefore  $s = fd_R(M, N) < \infty$ . Now by Theorem 5.2,  $s = depth R_{\mathfrak{p}} - depth_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - depth_{R_{\mathfrak{p}}} N_{\mathfrak{p}}$  for some  $\mathfrak{p} \in \text{Supp}(M) \cap \text{Supp}(N)$ . Now choose an integer  $n > CI fd_R M = Rfd_R M \geq depth R_{\mathfrak{p}} - depth_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \geq s$ . Therefore  $\text{Tor}_n^R(N, M) = 0$ .

(b) It follows easily from Corollary 5.7(b).  $\square$

## 6. THE FINITE CASE

In this section  $M$  is a finite  $R$ -module. We study the behavior of the new homological flat dimensions and especially of the (upper) Cohen-Macaulay flat dimension. Since for a finite  $R$ -module  $M$ , there is the equality  $\text{fd}_R M = \text{pd}_R M$  and  $\text{Gfd}_R M = \text{G-dim}_R M$  by [20], we have

$$\begin{aligned} \text{CM}^* \text{fd}_R M &= \inf\{\text{pd}_Q M' - \text{pd}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a CM}^*\text{-quasi-deformation}\}, \\ \text{CMfd}_R M &= \text{CM-dim}_R M, \text{G}^* \text{fd}_R M = \text{G}^* \text{-dim}_R M, \text{ and } \text{CI} \text{fd}_R M = \text{CI-dim}_R M. \end{aligned}$$

**Remark 6.1.** *It can be seen that if  $\text{CM}^* \text{fd}_R M < \infty$ , then there is the equality*

$$\text{CM}^* \text{fd}_R M + \text{depth}_R M = \text{depth } R.$$

Let  $\text{Syz}_n^R(M)$  to denote the  $n$ -th syzygy module of  $M$ . Then by an argument similar to that of [43, (2.5)] we have the following proposition:

**Proposition 6.2.** *For each  $n \geq 0$  there is the equality*

$$\text{CM}^* \text{fd}_R \text{Syz}_n^R(M) = \max\{\text{CM}^* \text{fd}_R M - n, 0\}.$$

**Theorem 6.3.** *The following conditions are equivalent:*

- (i) *The ring  $R$  is Cohen-Macaulay.*
- (ii)  *$\text{CM}^* \text{fd}_R M < \infty$  for every not necessarily finite  $R$ -module  $M$ .*
- (iii)  *$\text{CM}^* \text{fd}_R M < \infty$  for every finite  $R$ -module  $M$ .*
- (iv)  *$\text{CM}^* \text{fd}_R M = 0$  for every finite  $R$ -module  $M$  with  $\text{depth}_R M \geq \text{depth } R$ .*
- (v)  *$\text{CM}^* \text{fd}_R k = \text{depth } R$ .*
- (vi)  *$\text{CM}^* \text{fd}_R k < \infty$ .*

*Proof.* (i)  $\Rightarrow$  (ii) Let  $\hat{R}$  be the  $\mathfrak{m}$ -adic completion of  $R$ . Since  $R$  is Cohen-Macaulay, so is  $\hat{R}$ . Therefore by Cohen's structure theorem,  $\hat{R}$  is isomorphic to  $Q/J$ , where  $Q$  is a regular local ring. By Cohen-Macaulay-ness of  $\hat{R}$  and regularity of  $Q$ , the ideal  $J$  is perfect. Thus  $R \rightarrow \hat{R} \leftarrow Q$  is a  $\text{CM}^*$ -quasi-deformation. Since  $Q$  is regular  $\text{fd}_Q(M \otimes_R \hat{R})$  is finite. Thus  $\text{CM}^* \text{fd}_R M$  is finite.

(ii)  $\Rightarrow$  (iii) is trivial.

(iii)  $\Rightarrow$  (iv) follows by applying Remark 6.1 to the  $R$ -module  $M$ .

(iv)  $\Rightarrow$  (v) by [12, (1.3.7)] we have  $\text{depth}_R \text{Syz}_n^R(k) \geq \min(n, \text{depth } R)$ . In particular, if we choose  $n \geq \text{depth } R$  we get  $\text{CM}^* \text{fd}_R M = 0$ . Thus by Remark 6.1  $\text{CM}^* \text{fd}_R k = \text{depth } R$ .

(v)  $\Rightarrow$  (vi) is trivial.

(vi)  $\Rightarrow$  (i) follows from  $\text{CM-dim}_R k \leq \text{CM}^* \text{fd}_R k$  and [25, Theorem (3.9)].  $\square$

One can actually state similar theorems for upper Gorenstein flat and complete intersection flat dimensions.

As a consequence of the New Intersection Theorem of Peskine and Szpiro [34], Hochster [26] and P. Roberts [37] and [38] we have:

$$(*) \quad \text{cmd } R \leq \text{cmd}_R M.$$

The New Intersection Theorem is not true for CI-dimension,  $\text{G}^*$ -dimension, G-dimension, and CM-dimension, see Examples [41, (3.2)] and [44, (2.20)]. But the inequality  $(*)$  holds for  $\text{G}^*$ -dimension, see [41, (2.1)]. For G-dimension and CM-dimension we do not know whether the inequality  $(*)$  holds. However it holds for upper Cohen-Macaulay flat dimension as shown by the following theorem:

**Theorem 6.4.** *Let  $M$  be a finite  $R$ -module with finite upper Cohen-Macaulay dimension. Then  $\text{cmd } R \leq \text{cmd}_R M$ .*

*Proof.* Since  $\text{CM}^* \text{fd}_R M < \infty$ , there exists a  $\text{CM}^*$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$  such that  $\text{pd}_Q M' < \infty$ , where  $M' = M \otimes_R R'$ . From the surjectivity of  $Q \rightarrow R'$  we have  $\text{cmd}_Q M' = \text{cmd}_{R'} M'$ . Because  $R \rightarrow R'$  is a flat extension, the following (in)equalities hold:

$$\begin{aligned} \text{cmd } R + \text{cmd}_{R'} R'/\mathfrak{m}R' &= \text{cmd } R' \leq \text{cmd } Q \leq \text{cmd}_Q M' \\ &= \text{cmd}_R M + \text{cmd } R'/\mathfrak{m}R', \end{aligned}$$

where the first inequality holds by Lemma 3.6, and the second one is by the New Intersection Theorem. This gives us the desired inequality.  $\square$

**Corollary 6.5.** *If  $M$  is a Cohen-Macaulay module with  $\text{CM}^* \text{fd}_R M < \infty$ , then the base ring  $R$  is Cohen-Macaulay.*

## 7. HOMOLOGICAL INJECTIVE DIMENSIONS

It is well known that flat dimension and injective dimension are dual of each other. In particular there are the following equalities:

$$\text{fd}_R M^\vee = \text{id}_R M \text{ and } \text{id}_R M^\vee = \text{fd}_R M,$$

where  $M^\vee = \text{Hom}_R(M, E_R(k))$  and  $E_R(k)$  is the injective envelope of  $k$  over  $R$ . In this section we introduce dual of the complete intersection flat dimension and the Cohen-Macaulay flat dimension.

**Definition 7.1.** *Let  $M \neq 0$  be an  $R$ -module. The complete intersection injective dimension, upper Gorenstein injective dimension, and upper Cohen-Macaulay injective dimension of  $M$ , are defined as:*

$$\begin{aligned} \text{CI} \text{id}_R M &:= \inf \{ \text{id}_Q M' - \text{fd}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a CI-quasi-deformation} \} \\ G^* \text{id}_R M &:= \inf \{ \text{id}_Q M' - \text{fd}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a } G^* \text{-quasi-deformation} \} \\ \text{CM}^* \text{id}_R M &:= \inf \{ \text{id}_Q M' - \text{fd}_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a } \text{CM}^* \text{-quasi-deformation} \}, \end{aligned}$$

respectively. We complement this by  $\text{Hfd}_R 0 = -\infty$  for  $H = \text{CI}, G^*, \text{ and } \text{CM}^*$ .

In [40, Definition 2.6] Sather-Wagstaff introduced the upper complete intersection injective dimension of an  $R$ -module  $M$  as

$$\text{CI}^* \text{id}_R M := \inf \left\{ \text{id}_Q M' - \text{fd}_Q R' \mid \begin{array}{l} R \rightarrow R' \leftarrow Q \text{ is a CI-quasi-deformation} \\ \text{such that } R' \text{ has Gorenstein formal} \\ \text{fibre and } R'/\mathfrak{m}R' \text{ is Gorenstein} \end{array} \right\}.$$

We use the upper complete intersection injective dimension for a dual version of Theorem 4.5. The following theorem shows that the upper Cohen-Macaulay injective dimension characterizes Cohen-Macaulay local rings.

**Theorem 7.2.** *The following conditions are equivalent.*

- (i) *The ring  $R$  is Cohen-Macaulay.*
- (ii)  *$\text{CM}^* \text{id}_R M < \infty$  for every  $R$ -module  $M$ .*
- (iii)  *$\text{CM}^* \text{id}_R M < \infty$  for every finite  $R$ -module  $M$ .*
- (iv)  *$\text{CM}^* \text{id}_R k < \infty$ .*



*Proof.* (i)  $\Rightarrow$  (ii) Let  $\widehat{R}$  be the  $\mathfrak{m}$ -adic completion of  $R$ . Since  $R$  is Cohen-Macaulay, so is  $\widehat{R}$ . Therefore by Cohen's structure theorem,  $\widehat{R}$  is isomorphic to  $Q/J$ , where  $Q$  is a local regular ring. Hence due to Cohen-Macaulay-ness of  $\widehat{R}$  and regularity of  $Q$ , the ideal  $J$  is perfect. Thus  $R \rightarrow \widehat{R} \leftarrow Q$  is a  $\text{CM}^*$ -quasi-deformation. Since  $Q$  is regular  $\text{id}_Q(M \otimes_R \widehat{R})$  is finite, so  $\text{CM}^* \text{id}_R M$  is finite.

(ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (iv) are trivial.

(iv)  $\Rightarrow$  (i) Suppose  $\text{CM}^* \text{id}_R k < \infty$ . So that there exists a  $\text{CM}^*$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$ , such that  $\text{id}_Q(k \otimes_R R')$  is finite. It is clear that  $k \otimes_R R'$  is a finite  $Q$ -module. Consequently  $Q$  is a Cohen-Macaulay ring by the Bass Theorem. We plan to show that  $R'$  is a Cohen-Macaulay ring. Let  $I = \ker(Q \rightarrow R')$  which is perfect by definition. We have

$$\begin{aligned} \text{ht } I &= \text{grade}(I, Q) \\ &= \text{pd}_Q R' \\ &= \text{depth } Q - \text{depth}_Q R' \\ &= \text{depth } Q - \text{depth } R' \\ &= \dim Q - \text{depth } R' \\ &= \text{ht } I + \dim R' - \text{depth } R', \end{aligned}$$

in which the equalities follow from Cohen-Macaulay-ness of  $Q$ ; perfectness of  $I$ ; Auslander-Buchsbaum formula; [12, (1.2.26)]; Cohen-Macaulay-ness of  $Q$ ; and [33, Page 250] respectively. Therefore we obtain that  $\dim R' - \text{depth } R' = 0$ , that is  $R'$  is Cohen-Macaulay. Now [12, Theorem (2.1.7)] gives us the desired result.  $\square$

In the same way one can show that the upper Gorenstein injective dimension detects the Gorenstein property and the complete intersection injective dimension detects the complete intersection property of local rings.

The proof of the above theorem says some thing more, viz., a local ring  $R$  is Cohen-Macaulay if and only if there exists a finite  $R$ -module of finite upper Cohen-Macaulay injective dimension. In other words every finite  $R$ -module is a test module for the Cohen-Macaulay property of a local ring. So we state the following corollary, which is analogous to the definition of a Gorenstein ring:

**Corollary 7.3.** *A local ring  $R$  is Cohen-Macaulay if and only if  $\text{CM}^* \text{id}_R R < \infty$ .*

In [24, Theorem (4.5)] Foxby and Frankild proved that if a ring admits a cyclic module of finite Gorenstein injective dimension, then the base ring is Gorenstein. Parallel to their result we have the the following proposition.

**Proposition 7.4.** *If  $G^* \text{id}_R C < \infty$  for a cyclic  $R$ -module  $C$ , then  $R$  is a Gorenstein local ring.*

*Proof.* There is a  $G^*$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$  such that  $\text{id}_Q(C \otimes_R R')$  is finite. Since  $C \otimes_R R'$  is a cyclic  $R'$ -module and  $R'$  is a cyclic module over  $Q$ , we see that  $C \otimes_R R'$  is a cyclic module over  $Q$ . So that  $Q$  is Gorenstein by [35]. Hence  $R'$  is a Gorenstein ring because the kernel of  $Q \rightarrow R'$  is a Gorenstein ideal. Consequently  $R$  is Gorenstein.  $\square$

**Lemma 7.5.** *There is an equality*

$$\text{Hid}_R M = \inf \left\{ \text{id}_Q M' - \text{fd}_Q R' \mid \begin{array}{l} R \rightarrow R' \leftarrow Q \text{ is an } H\text{-quasi-deformation such} \\ \text{that the closed fibre of } R \rightarrow R' \text{ is artinian} \end{array} \right\},$$

for  $H = CI^*$ ,  $CI$ ,  $G^*$ , and  $CM^*$ .

*Proof.* We prove the lemma for  $H = CM^*$  only since the other cases are similar. Let  $R \rightarrow R' \leftarrow Q$  be an  $CM^*$ -quasi-deformation. Choose  $\mathfrak{p} \in \text{Spec}(R')$  such that it is a minimal prime ideal containing  $\mathfrak{m}R'$ ; thus  $\mathfrak{m} = \mathfrak{p} \cap R$  and  $\mathfrak{p} = \mathfrak{q}/J$  for some  $\mathfrak{q} \in \text{Spec}(Q)$ , where  $J = \ker(Q \rightarrow R')$ . Now the diagram  $R \rightarrow R'_\mathfrak{p} \leftarrow Q_\mathfrak{q}$  is a  $CM^*$ -quasi-deformation. It is clear that  $\text{pd}_Q R' = \text{pd}_{Q_\mathfrak{q}} R'_\mathfrak{p}$ . Also we have

$$\text{id}_{Q_\mathfrak{q}}(M \otimes_R R'_\mathfrak{p}) = \text{id}_{Q_\mathfrak{q}}(M \otimes_R (R' \otimes_Q Q_\mathfrak{q})) = \text{id}_{Q_\mathfrak{q}}(M' \otimes_Q Q_\mathfrak{q}) \leq \text{id}_Q M' < \infty.$$

Hence  $CM^* \text{id}_R M \leq \text{id}_{Q_\mathfrak{q}}(M \otimes_R R'_\mathfrak{p}) - \text{pd}_{Q_\mathfrak{q}} R'_\mathfrak{p}$ . So the proof is complete.  $\square$

Recall that  $\text{width}_R M = \inf\{i \mid \text{Tor}_i^R(M, k) \neq 0\}$ . It is the dual notion for  $\text{depth}_R M$ . In particular by [15, (4.8)] we have  $\text{width}_R M = \text{depth}_R \text{Hom}_R(M, E_R(k))$ , where  $E_R(k)$  denote for injective envelope of  $k$  over  $R$ .

The *Chouinard injective dimension* is denoted by  $\text{Chid}_R M$  and is defined as,

$$\text{Chid}_R M := \sup\{\text{depth}_{R_\mathfrak{p}} M_\mathfrak{p} - \text{width}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

**Remark 7.6.** *We do not introduce Chouinard flat dimension because it coincides with  $R\text{fd}$  since by [15, (2.4)(b)], we have*

$$R\text{fd}_R M := \sup\{\text{depth}_{R_\mathfrak{p}} M_\mathfrak{p} - \text{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \mid \mathfrak{p} \in \text{Spec}(R)\}.$$

It is proved in [14] that for an  $R$ -module  $M$ ,  $\text{Chid}_R M$  is a refinement of  $\text{id}_R M$ , that is

$$\text{Chid}_R M \leq \text{id}_R M,$$

with equality if  $\text{id}_R M$  is finite.

In the Theorem 7.11 we partly extend this relation for our homological injective dimensions. As of the writing of this paper, the authors do not know if the equality holds in general. Before stating the theorem we need some lemmas.

**Lemma 7.7.** *Suppose that  $Q \rightarrow S$  is a surjective local homomorphism and  $N$  is an  $S$ -module. Then we have*

$$\text{width}_S N = \text{width}_Q N.$$

*Proof.* We have the following equalities:

$$\begin{aligned} \text{width}_S N &= \text{depth}_S \text{Hom}_S(N, E_S(k)) \\ &= \text{depth}_S \text{Hom}_S(N, \text{Hom}_Q(S, E_Q(k))) \\ &= \text{depth}_S \text{Hom}_Q(N, E_Q(k)) \\ &= \text{depth}_Q \text{Hom}_Q(N, E_Q(k)) \\ &= \text{width}_Q N, \end{aligned}$$

where the first one is by [15, (4.8)]; the second one is by [13, (10.1.15)]; the third one is by adjointness of  $\text{Hom}$  and tensor; the fourth one is true since  $Q \rightarrow S$  is surjective; while the last one is again by [15, (4.8)]. Here we used  $k$  for the residue fields of  $Q$  and  $S$ , and  $E_Q(k)$  and  $E_S(k)$  for the injective envelopes of  $k$  over respectively  $Q$  and  $S$ .  $\square$

Dualizing the proof of Proposition 3.2 and using the above lemma one easily shows

**Proposition 7.8.** *Let  $Q \rightarrow S$  be a CM-deformation, and  $N$  be an  $S$ -module. Then there is the equality:*

$$\text{Chid}_S N + G\text{-dim}_Q S = \text{Chid}_Q N.$$

**Lemma 7.9.** *Suppose that  $(R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, l)$  is a local ring homomorphism, and  $M$  is an  $R$ -module. Then we have*

$$\text{width}_S(M \otimes_R S) = \text{width}_R M.$$

*Proof.* Let  $F_M$  be a flat resolution of  $M$  over  $R$ . Therefore  $F_M \otimes_R S$  is a flat resolution of  $M \otimes_R S$  over  $S$ . So we have

$$\begin{aligned} \text{width}_S(M \otimes_R S) &= \inf\{i \mid \text{Tor}_i^S(M \otimes_R S, l) \neq 0\} \\ &= \inf\{i \mid \text{H}_i((F_M \otimes_R S) \otimes_S l) \neq 0\} \\ &= \inf\{i \mid \text{H}_i(F_M \otimes_R l) \neq 0\} \\ &= \inf\{i \mid \text{Tor}_i^R(M, l) \neq 0\} \\ &= \inf\{i \mid \text{Tor}_i^R(M, k) \neq 0\} \\ &= \text{width}_R M. \end{aligned}$$

□

**Lemma 7.10.** *Let  $R \rightarrow S$  be a flat local homomorphism and let  $M$  be an  $R$ -module. Then*

$$\text{Chid}_R M \leq \text{Chid}_S(M \otimes_R S).$$

*Proof.* Let  $\mathfrak{p} \in \text{Spec}(R)$  such that  $\text{Chid}_R M = \text{depth } R_{\mathfrak{p}} - \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ . Let  $\mathfrak{q} \in \text{Spec}(S)$  contain  $\mathfrak{p}S$  minimally. Since  $R \rightarrow S$  is a flat local homomorphism we have  $\mathfrak{p} = \mathfrak{q} \cap R$  and  $\text{ht } \mathfrak{p} = \text{ht } \mathfrak{q}$ . Hence:

$$\begin{aligned} \text{Chid}_R M &= \text{depth } R_{\mathfrak{p}} - \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &= \text{depth } S_{\mathfrak{q}} - \text{width}_{S_{\mathfrak{q}}}(M_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{q}}) \\ &= \text{depth } S_{\mathfrak{q}} - \text{width}_{S_{\mathfrak{q}}}(M \otimes_R S)_{\mathfrak{q}} \\ &\leq \text{Chid}_S(M \otimes_R S), \end{aligned}$$

in which the second equality holds by Lemma 7.9 and the fact that  $R_{\mathfrak{p}} \rightarrow S_{\mathfrak{q}}$  has artinian closed fibre. □

**Theorem 7.11.** *Suppose that  $M$  is an  $R$ -module such that  $\text{Hid}_R M < \infty$  for  $H = CI^*$ ,  $CI$ ,  $G^*$ , or  $CM^*$ . Then there is the inequality*

$$\text{Chid}_R M \leq \text{Hid}_R M,$$

and if  $M$  is a finite module we have

$$\text{Chid}_R M = \text{Hid}_R M = \text{depth } R.$$

*Proof.* We prove the theorem for  $H = CM^*$  and the other cases are similar. Choose by Lemma 7.5 a  $CM^*$ -quasi-deformation  $R \rightarrow R' \leftarrow Q$ , such that  $CM^*id_R M = id_Q M' - fd_Q R'$ , where  $M' = M \otimes_R R'$ , and the closed fibre of  $R \rightarrow R'$  is artinian. Hence we have

$$\begin{aligned} CM^*id_R M &= id_Q M' - fd_Q R' \\ &= Chid_Q M' - fd_Q R' \\ &= Chid_{R'} M' \geq Chid_R M, \end{aligned}$$

in which the second equality comes by [14], and the third one by Proposition 7.8; while the inequality is by Lemma 7.10.

Now let  $M$  be a finite  $R$ -module, therefore  $M'$  is a finite  $Q$ -module. So by the Bass Theorem [33, (18.9)], and the Auslander-Buchsbaum formula, and the fact that the closed fibre of  $R \rightarrow R'$  is artinian we have:

$$CM^*id_R M = id_Q M' - fd_Q R' = \text{depth } Q - \text{depth } Q + \text{depth } R' = \text{depth } R' = \text{depth } R.$$

On the other hand, since  $CM^*id_R M < \infty$  and  $M$  is finite,  $R$  is a Cohen-Macaulay ring. Thus using [39, (3.6)] we see that  $Chid_R M = \text{depth } R$ . Hence  $CM^*id_R M = Chid_R M = \text{depth } R$ .  $\square$

**Corollary 7.12.** *Let  $M$  be an  $R$ -module. Then we have the following chain of inequalities:*

$$Chid_R M \leq CM^*id_R M \leq G^*id_R M \leq CIid_R M \leq CI^*id_R M \leq id_R M,$$

*with equality to the left of any finite number for finite modules or, if  $id_R M < \infty$  for arbitrary module  $M$ .*

Now we prove the dual result of Theorem 4.5.

**Theorem 7.13.** *Suppose that  $R$  has a dualizing complex  $D$ , and  $M$  is an  $R$ -module. Then there is an inequality*

$$Gid_R M \leq CI^*id_R M.$$

*Proof.* We can actually assume that  $CI^*id_R M$  is finite. So that by [40, Proposition 3.5], there exists a CI-quasi-deformation  $R \rightarrow R' \leftarrow Q$  such that  $Q$  is complete, the closed fibre  $R'/\mathfrak{m}R'$  is artinian and Gorenstein and  $id_Q(M \otimes_R R')$  is finite. Therefore by Remark 2.4(b),  $M \otimes_R R'$  belongs to the Bass class  $\mathbf{B}(Q)$ . By the same argument as in the proof of Theorem 4.5 one can show that  $M \otimes_R R'$  belongs to the Bass class  $\mathbf{B}(R')$ . Consequently Remark 2.4(b), gives us that  $Gid_{R'}(M \otimes_R R') < \infty$ . Since  $R'/\mathfrak{m}R'$  is a Gorenstein local ring, by [8, Proposition 4.2], we have  $R \rightarrow R'$  is a Gorenstein local homomorphism. Therefore by [8, Theorem 5.1], the complex  $D \otimes_R^{\mathbf{L}} R'$  is a dualizing complex of  $R'$ . Consequently [16, Theorem 5.3] gives finiteness of  $Gid_R M$ . Hence using [16, Theorem 6.8] we have  $Gid_R M = Chid_R M \leq CI^*id_R M$  as desired.  $\square$

By the above theorem we have

$$Chid_R M \leq Gid_R M \leq CI^*id_R M \leq id_R M.$$

Now we define a Cohen-Macaulay injective dimension to complete this sequence of inequalities. Notice that there is a notion of Cohen-Macaulay injective dimension in [28] which is different with ours.

**Definition 7.14.** Let  $M \neq 0$  be an  $R$ -module. The Cohen-Macaulay injective dimension of  $M$ , is defined by ,

$$CMid_R M := \inf\{Gid_Q M' - Gfd_Q R' \mid R \rightarrow R' \leftarrow Q \text{ is a CM-quasi-deformation}\}.$$

We complement this by  $CMid_R 0 = -\infty$ .

Therefore by taking the trivial CM-quasi-deformation  $R \rightarrow R \leftarrow R$ , one has  $CMid_R M \leq Gid_R M$ . Suppose that  $R$  has a dualizing complex. Then by [16, Proposition 5.5] one has  $Gid_{R_p} M_p \leq Gid_R M$  for each prime ideal  $p$  of  $R$ . So that by the proof of Lemma 7.5, one has

$$CMid_R M = \inf \left\{ \left. \begin{array}{l} Gid_Q M' - Gfd_Q R' \\ R \rightarrow R' \leftarrow Q \text{ is a CM-quasi-deformation} \\ \text{such that the closed fibre of} \\ R \rightarrow R' \text{ is artinian} \end{array} \right\} \right\}.$$

Therefore as in Theorem 7.11 one can show that  $Chid_R M \leq CMid_R M$ . Hence when the ring  $R$  admits a dualizing complex, then there is the following sequence of inequalities

$$Chid_R M \leq CMid_R M \leq Gid_R M \leq CI^*id_R M \leq id_R M,$$

with equality to the left of any finite number for finite modules or, if  $id_R M$  or  $Gid_R M$  if finite, for arbitrary module  $M$ .

**Theorem 7.15.** The following conditions are equivalent.

- (i) The ring  $R$  is Cohen-Macaulay.
- (ii)  $CMid_R M < \infty$  for every  $R$ -module  $M$ .
- (iii)  $CMid_R M < \infty$  for every finite  $R$ -module  $M$ .
- (iv)  $CMid_R k < \infty$ .

*Proof.* (i)  $\Rightarrow$  (ii) It follows by the inequality  $CMid_R M \leq CM^*id_R M$  and Theorem 7.2.

(ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (iv) are trivial.

(iv)  $\Rightarrow$  (i) Suppose  $CM^*id_R k < \infty$ . So there exists a CM-quasi-deformation  $R \rightarrow R' \leftarrow Q$ , such that  $Gid_Q(k \otimes_R R')$  is finite. Since  $k \otimes_R R'$  is a cyclic module over  $Q$ , we see that  $Q$  is a Gorenstein ring by [24, Theorem 4.5]. The rest of proof is the same as that of Theorem 7.2.  $\square$

## 8. THE AUSLANDER-BUCHSBAUM FORMULA

In this section we give a necessary and sufficient condition for our homological flat dimensions to satisfy a formula of Auslander-Buchsbaum type. Our main result is Theorem 8.4 below. Recall that  $\text{grade}(\mathfrak{p}, M) = \inf\{i \mid \text{Ext}_R^i(R/\mathfrak{p}, M) \neq 0\}$ .

**Lemma 8.1.** Let  $R$  be a local ring, and  $M$  an  $R$ -module of finite depth. Then  $\text{depth}_R M \leq \text{depth}_{R_p} M_p + \dim R/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Spec}(R)$  if and only if  $\text{depth}_R M \leq \text{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$  for all  $\mathfrak{p} \in \text{Spec}(R)$ .

*Proof.* The only if part is trivial. For the if part we show that for modules  $M$  and  $N$  with  $N$  finite we have  $\text{Ext}_R^i(N, M) = 0$  for  $i < \text{depth}_R M - \dim N$ . We do this by induction on  $\dim N$ . If  $\dim N = 0$ , then  $N$  has finite length, and in this case an easy induction proves the result. Now let  $\dim N = t$ . By a method similar to that of [33, (17.1)] it is sufficient to take  $N = R/\mathfrak{p}$  such that  $\dim R/\mathfrak{p} = t$ . Let  $s < \text{depth}_R M - t \leq \text{depth}_{R_p} M_p$ . We have to show that  $E = \text{Ext}_R^s(R/\mathfrak{p}, M) = 0$ . If  $E \neq 0$ , there is a non-zero element  $e \in E$ . Since  $E_p = 0$  there is an element

$u \in R \setminus \mathfrak{p}$  such that  $ue = 0$ . Now the exact sequence  $0 \rightarrow R/\mathfrak{p} \xrightarrow{u} R/\mathfrak{p} \rightarrow N' \rightarrow 0$ , gives rise the exact sequence

$$\mathrm{Ext}_R^s(N', M) \rightarrow \mathrm{Ext}_R^s(R/\mathfrak{p}, M) \xrightarrow{u} \mathrm{Ext}_R^s(R/\mathfrak{p}, M),$$

in which the left most module equal to zero by the induction hypothesis. So  $u$  is injective and therefore  $e = 0$ , which is a contradiction.  $\square$

**Proposition 8.2.** *Let  $M$  be an  $R$ -module such that  $\mathrm{Rfd}_R M + \mathrm{depth}_R M = \mathrm{depth} R$ . Then*

$$\mathrm{depth}_R M \leq \mathrm{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$$

for all  $\mathfrak{p} \in \mathrm{Spec}(R)$ . The converse is true over Cohen-Macaulay rings.

*Proof.* Let  $\mathfrak{p} \in \mathrm{Spec}(R)$  be an arbitrary prime ideal. Therefore we have

$$\mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Rfd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \mathrm{Rfd}_R M = \mathrm{depth} R - \mathrm{depth}_R M.$$

So that

$$\begin{aligned} \mathrm{depth}_R M &\leq \mathrm{depth} R - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p}. \end{aligned}$$

Now from Lemma 8.1 we obtain that  $\mathrm{depth}_R M \leq \mathrm{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$  for all  $\mathfrak{p} \in \mathrm{Spec}(R)$ .

Next suppose that  $R$  is a Cohen-Macaulay ring. Choose a prime ideal  $\mathfrak{p} \in \mathrm{Spec}(R)$  such that  $\mathrm{Rfd}_R M = \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$ . Then from the hypothesis and [33, Page 250] we have:

$$\begin{aligned} \mathrm{Rfd}_R M &= \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &= \dim R_{\mathfrak{p}} - \mathrm{depth}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \\ &\leq \dim R - \dim R/\mathfrak{p} - \mathrm{grade}(\mathfrak{p}, M) \\ &\leq \dim R - \mathrm{depth}_R M \\ &= \mathrm{depth} R - \mathrm{depth}_R M \\ &\leq \mathrm{Rfd}_R M, \end{aligned}$$

which completes the proof.  $\square$

Combining Proposition 8.2, and [15, Theorem (3.4)] we have

**Corollary 8.3.** *Let  $R$  be a ring and  $M$  be an  $R$ -module. The following then are equivalent:*

- (i)  $\mathrm{Rfd}_R M + \mathrm{depth}_R M = \mathrm{depth} R$  for every  $R$ -module  $M$  of finite depth.
- (ii)  $R$  is a Cohen-Macaulay ring and  $\mathrm{depth}_R M \leq \mathrm{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$  for every  $R$ -module  $M$  of finite depth, and for all  $\mathfrak{p} \in \mathrm{Spec}(R)$ .

This is an extension of [15, Theorem (3.4)].

Now we state the main result of this section.

**Theorem 8.4.** *Let  $R$  be a Cohen-Macaulay local ring and let  $M$  be an  $R$ -module of finite  $\mathrm{Hfd}_R M$  for  $H = CI, G^*, CM^*$ , and  $CM$ . Then  $\mathrm{Hfd}_R M + \mathrm{depth}_R M = \mathrm{depth} R$  if and only if  $\mathrm{depth}_R M \leq \mathrm{grade}(\mathfrak{p}, M) + \dim R/\mathfrak{p}$  for all  $\mathfrak{p} \in \mathrm{Supp}(M)$ .*

Dual to the Proposition 8.2 one can prove the following.

**Proposition 8.5.** *Let  $R$  be a local ring, and  $M$  an  $R$ -module such that  $\text{Chid}_R M + \text{depth}_R M = \text{depth } R$ . Then*

$$\text{width}_R M \leq \text{width}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} + \dim R/\mathfrak{p}$$

*for all  $\mathfrak{p} \in \text{Spec}(R)$ . The converse is true over Cohen-Macaulay rings.*

### ACKNOWLEDGMENT

The authors would like to thank H. B. Foxby and S. Sather-Wagstaff for their useful comments, and Mehrdad Shahshahani for his careful reading of this paper.

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